Optimal lower bound of quantum uncertainty from extractable classical information

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The sum of entropic uncertainties for the measurement of two non-commuting observables is not always reduced by the amount of entanglement (quantum memory) between two parties, and in certain cases may be impacted by quantum correlations beyond entanglement (discord). However, in any operational situation, the optimal lower bound of entropic uncertainty is given by fine-graining. Here we derive a new uncertainty relation where the maximum possible reduction of uncertainty is shown to be given by the *extractable classical information*, thus providing an explanation in terms of physical resources for the operationally relevant fine-graining for determining the minimum uncertainty. We illustrate this result through examples of pure and mixed entangled states, and also separable states with non-vanishing discord. Using our uncertainty relation we further show that even in the absence of any quantum correlations between the two parties, the sum of uncertainties may be reduced with the help of classical correlations.

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A fundamental difference from classical theory is that quantum theory limits the precision of the measurement outcomes for the measurement of two non-commuting observables. This quantum feature which was first introduced by Heisenberg [1] is given by

$$\Delta R.\Delta S \geq \frac{1}{2}|\langle [R,S]\rangle_\rho|, \tag{1}$$

where the uncertainty of the observable $\alpha \in \{R, S\}$ measured on the system in the state ρ is captured in terms of standard deviation $\Delta\alpha$, and was then extended by Robertson [2] for more general observables. In order to overcome certain drawbacks of the Eq.(1), such as the state-dependence of its right hand side, the uncertainty relation was recast in the entropic form where the uncertainty is measured by the Shannon entropy [3]. The entropic uncertainty relation (EUR) was first introduced by Deutsch [4]. An improved form of EUR given by

$$\mathcal{H}(R) + \mathcal{H}(S) \geq \log_2 \frac{1}{c}, \tag{2}$$

where $\mathcal{H}(i)$ represents the Shannon entropy for the measurement of observable $i \in \{R, S\}$, and the complementarity of the observables R and S is quantified by the quantity c $(= \max_{i,j} |\langle r_i|s_j\rangle|^2$, with $|r_i\rangle$ and $|s_j\rangle$ the eigenvectors of R and S, respectively), was first conjectured in Ref.[5] and then proved in Ref.[6].

Considering the correlation of the observed system with another system called quantum memory, Berta et al. [7] have modified the lower bound of entropic uncerrtainty. The modified form of EUR in the presence of quantum memory is given by [7]

$$\mathcal{S}(R_A|B) + \mathcal{S}(S_A|B) \geq \log_2 \frac{1}{c} + \mathcal{S}(A|B) \tag{3}$$

where the uncertainty for the measurement of the observable R_A (S_A) on Alice's system (labeled by 'A') by accessing the information stored in the quantum memory with Bob (labelled by 'B') is measured by $\mathcal{S}(R_A|B)$ ($\mathcal{S}(S_A|B)$) which is the conditional von Neumann entropy of the state given by

$$\begin{aligned}\rho_{R_A(S_A)B} &= \sum_j (|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)\rho_{AB}(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I) \\ &= \sum_j p_j^{R_A(S_A)} \Pi_j^{R_A(S_A)} \otimes \rho_{B|j}^{R_A(S_A)},\end{aligned} \tag{4}$$

where $\Pi_j^{R_A(S_A)}$'s are the orthogonal projectors on the eigenstate $|\psi_j\rangle_{R_A(S_A)}$ of observable $R_A(S_A)$, $p_j^{R_A(S_A)} = Tr[(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)\rho_{AB}(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)]$, $\rho_{B|j}^{R_A(S_A)} = Tr_A[(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)\rho_{AB}(|\psi_j\rangle_{R(S)}\langle\psi_j| \otimes I)]/p_j^{R_A(S_A)}$ and ρ_{AB} is the state of joint system 'A' and 'B'. EUR in presence of quantum memory is modified by the quantity $\mathcal{S}(A|B)$ $(= \mathcal{S}(\rho_{AB}) - \mathcal{S}(\rho_B)$, where $\rho_B = Tr_A[\rho_{AB}]$) which measures the amount of one-way distilable entanglement [8]. For shared maximal entanglement (i.e., $S(A|B) = -1$) between the system and the memory, there is no uncertainty in the measurement of incompatible observables. EUR in the presence of quantum memory has been brought out in two recent experiments using respectively, pure [9] and mixed states [10]. For experimental purposes, one can obtain the uncertanty relation given by [10] (with the help of Fano's inequality [11])

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \log_2(\frac{1}{c}) + \mathcal{S}(A|B), \tag{5}$$

where p_d^R (p_d^S) is the probability of getting different outcomes when Alice and Bob measure the same observables R (S) on their respective system. Here the lower bound of the sum of uncertainties for the shared state ρ_{AB} is given by

$$\mathcal{L}_1(\rho_{AB}) = \log_2(\frac{1}{c}) + \mathcal{S}(A|B) \tag{6}$$

In a following work, Pati et al. [12] have derived a tighter lower bound of the uncertainty relation using the state $\rho_{R_A(S_A)B}$ given by Eq.(4) to be

$$\mathcal{S}(R_A|B)+\mathcal{S}(S_A|B) \geq \log_2\frac{1}{c}+\mathcal{S}(A|B) + \max\{0, D_A(\rho_{AB})-C_A^M(\rho_{AB})\}, \tag{7}$$

where the quantum discord $D_A(\rho_{AB})$ is given by [13, 14]

$$D_A(\rho_{AB})=\mathcal{I}(\rho_{AB})-C_A^M(\rho_{AB}), \tag{8}$$

with $\mathcal{I}(\rho_{AB})$ $(=\mathcal{S}(\rho_A)+\mathcal{S}(\rho_B)-\mathcal{S}(\rho_{AB}))$ being the mutual information of the state ρ_{AB} which contains the total correlation present in the state ρ_{AB} shared between the system A and the system B, and the classical information $C_A^M(\rho_{AB})$ for the shared state ρ_{AB} (when Alice measures on her system) is given by

$$C_A^M(\rho_{AB})=\max_{\Pi^{R_A}}[\mathcal{S}(\rho_B)-\sum_{j=0}^{1}p_j^{R_A}\mathcal{S}(\rho_{B|j}^{R_A})] \tag{9}$$

In this case, the lower bound of the sum of Bob's uncertainty about Alice's measurement outcome for the measurement of observable R and S is given by

$$\mathcal{L}_2(\rho_{AB}) = \log_2\frac{1}{c}+\mathcal{S}(A|B) + \max\{0, D_A(\rho_{AB})-C_A^M(\rho_{AB})\}, \tag{10}$$

which becomes a tighter lower bound comapred to $\mathcal{L}_1$ given by Eq.(6) for those state whose quantum discord is larger than the classical information, which is true for example, for a class of states including Werner states and isotropic states.

In another recent work [15], we have shown that the lower bound of the uncertainty relation given by Eqs. (3) and (7) are not *optimal* in an operational sense, as illustrated by the analysis of an experiment using mixed states [10]. We have obtained the *optimal* lower bound of entropic uncertainty using fine-frained uncertainty relation [16]. For an experimental situation [10] where Alice and Bob both measure the same observable on their system, we have derived a new uncertainty relation that captures the optimal lower bound for Bob's uncertainty about Alice's measurement outcomes. Our uncertainty relation is given by [15]

$$\mathcal{H}(p_d^R)+\mathcal{H}(p_d^S)\geq\mathcal{H}(p_d^{\sigma_z})+\mathcal{H}(p_{\inf}^S), \tag{11}$$

where the lower bound given by

$$\mathcal{L}_3(\rho_{AB})=\mathcal{H}(p_d^{\sigma_z})+\mathcal{H}(p_{\inf}^S) \tag{12}$$

is operationally optimal. $\mathcal{L}_3(\rho_{AB})$ is obtained with the help of the fine-grained uncertainty relation [16] which gives the infimum winning probability $p_d^{\sigma_z}$ $(p_{\inf}^S)$ (corresponding to the minimum uncertainty) for the measurement of observable σ_z ($\sigma_S=\vec{n}_S.\vec{\sigma}\neq\sigma_z$ with $\vec{n}_S\equiv\{\sin(\theta_S)\cos(\phi_S),\sin(\theta_S)\sin(\phi_S),\cos(\theta_S)\}$ being a unit vector and $\vec{\sigma}=\{\sigma_x,\sigma_y,\sigma_z\}$ are the Pauli matrices) corresponding to the game ruled by the winning condition given by [15]

$$\begin{aligned} V(a,b) &= 1 \qquad \text{iff } a\oplus b=1, \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{13}$$

where 'a' and 'b' are the binary outcomes (i.e., $\{a,\ b\}\in\{0,1\}$) for Alice and Bob, respectively. $V(a,b)$ captures the experimental situation used in the Ref.[10], i.e., Alice and Bob measure the same observables and calculate the probability of getting different outcomes. In measurements and communication involving two parties, the lower bound of entropic uncertainty cannot fall below the bound (12).

The motivation of the present work is to understand the above result from a deeper physical perspective, *i.e.*, we aim to provide an explanation in terms of physical resources responsible for the optimal reduction of entropic uncertainty, which is given operationally by using the fine-grained uncertainty relation. In other words, we investigate the question as to which physical quantity is responsible for reduction of the uncertainty optimally in an experimental situation involving the measurement of two incompatible observables in the presence of shared states (correlations) between two parties. In [15] we have shown that the magnitude of quantum entanglement $S(A|B)$ does not exactly match with the optimal reduction of uncertainty in all situations. In [12] a combination of quantum discord and classical correlations, *viz.*, $\max\{0, D_A(\rho_{AB})-C_A^M(\rho_{AB})\}$ was introduced as a measure for obtaining a tighter lower bound compared to that obtained by just entanglement [7]. However, as we have shown [15], even such a combination fails to account in all cases for the operationally valid optimal lower bound given by fine-graining. In the present work we introduce the measure *extractable classical information* which as we show, contributes exactly to reducing the uncertainty by an amount leading to the optimal lower bound. Here, we derive a new uncertainty relation in terms of the extractable classical information. We illustrate our derived uncertainty relation with several examples of entangled, separable as well as classical states. It further follows from our relation that even in the absence of quantum correlations between the two parties, the uncertainty may be reduced with the help of classical correlations.

To derive the sum of uncertainties for the measurement of two incompatible observables R and S, we consider the following memory game [7]. In this game Bob prepares a particle (labeled by 'A') in a particular state, say, ρ_A and sends it to Alice who measures an observable chosen from the non-commuting set $\{R_A, S_A\}$ and communicates only the choice of the observable to Bob. Bob's task is to reduce his uncertainty about the Alice's measurement outcome. To win the game, Bob chooses

one of the following two strategies – (i) *classical strategy*; (ii) *quantum strategy.*

Classical strategy : Here, Bob prepares two particles (say, 1st particle labeled by A and 2nd particles labeled by B) in the identical state, $\rho = \rho_A = \rho_B$. The combined state of two particles is given by

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{14}$$

After preparation, Bob sends the 1st particle to Alice. When Alice communicates about her choice of measurement from the set of observables $\{R_A, S_A\}$, Bob measures the same observable on the 2nd particle possessed by himself. He infers about the Alice's measurement outcome from his own measurement outcome. Here, the uncertainty relation prevents Bob to know with arbitrary precision the measurement outcomes of two non-commuting observables. The EUR which gives the lower bound for the measurement of the above two non-commuting observables follows from Eq.(3) for product states and is given by [7, 17]

$$\mathcal{H}(R_B) + \mathcal{H}(S_B) \geq \log_2 \frac{1}{c} + \mathcal{S}(\rho_B), \tag{15}$$

where the subscript B labels Bob's measurement. Hence, Bob can not reduce his uncertainty about Alice's measurement outcome below the lower bound $\mathcal{L}_0(\rho_{AB})$ given by

$$\mathcal{L}_0(\rho_{AB}) = \log_2 \frac{1}{c} + \mathcal{S}(\rho_B). \tag{16}$$

Note that, the state given by Eq.(14) has zero classical correlation (i.e., $C_A^M = 0$) and zero quantum correlation (i.e., $D_A = 0$) [18].

Quantum strategy : In this strategy, Bob prepares two particles in a correlated state, ρ_{AB}, and sends the 1st particle to Alice and keeps the 2nd particle. To reduce his uncertainty further from the bound $\log_2 \frac{1}{c} + \mathcal{S}(\rho_B)$ (which is the lower bound of uncertainty corresponding to the *classical strategy*), Bob uses the correlations (quantum and/or classical) present in the state ρ_{AB}. After getting information about the choice of measurement, Bob measures the same observable as Alice's choice. Since, Alice and Bob measure independently on their respective systems, the order of measurement, i.e., who measures first, does not affect in the considered game. Here consider Alice communicates about her choice of observable from the set $\{R, S\}$ to Bob. First Bob measures the observable and then Alice measures. After the measurement performed by Bob with the observable that communicated by Alice, the combined state $\rho_{AR_B(S_B)}$ is given by

$$\rho_{AR_B(S_B)} = \sum_j p_j^{R_B(S_B)} \rho_{A|j}^{R_B(S_B)} \otimes \Pi_j^{R_B(S_B)}, \tag{17}$$

where $\Pi_j^{R_B(S_B)} = |\psi\rangle_{R_B(S_B)}\langle\psi|$, $\rho_{A|j}^{R_B(S_B)}$ $(= Tr_B[(I \otimes \Pi_j^{R_B(S_B)})\rho_{AB}])$ is the Alice's conditional state when Bob gets j-th outcome and $p_j^{R_B(S_B)}$ $(= Tr[(I \otimes |\psi\rangle_{R_B(S_B)}\langle\psi|)\rho_{AB}])$ is the probability of getting j-th outcome by Bob.

The classical information $C_B^M(\rho_{AB})$, given by

$$C_B^M(\rho_{AB}) = \max_{\Pi^{R_B}} [\mathcal{S}(\rho_A) - \sum_{j=0}^{1} p_j^{R_B} \mathcal{S}(\rho_{A|j}^{R_B})], \tag{18}$$

where $\rho_A = Tr_B[\rho_{AB}]$, gives the *maximum information* that Bob can extract on average about the Alice's system by measuring on his system when they share the state ρ_{AB}. Now, one may ask the following questions – what information can Bob extract about Alice's measurement outcomes? $C_B^M(\rho_{AB})$ contains the information about the Alice's measurement outcomes when she measures along a particular direction which maximizes the quantity $C_B(\rho_{AB})$ (where $C_B(\rho_{AB})$ is taken with out maximization in Eq.(18)). When Bob gets the j-th outcome for the measurement of the observable R_B on his system, his knowledge about Alice's measurement outcomes for the measurement in the eigenbasis of $\rho_{A|j}^{R_B}$ is given by the quantity $\mathcal{S}(\rho_{A|j}^{R_B})$. Since $\mathcal{S}(\rho_A)$ is Bob's uncertainty about Alice's outcome in the absence of correlations, from the Eq.(18), it can be easily seen that $C_B^M(\rho_{AB})$ measures the amount of Bob's uncertainty about Alice's measurement outcome reduced due to Bob's measurement. For example, for the shared Werner state ρ_{AB}^W [19] between Alice and Bob given by

$$\rho_{AB}^W = \frac{1-p}{4} I \otimes I + p|\psi^-\rangle\langle\psi^-|, \tag{19}$$

where I is the $(2 \otimes 2)$ unitary matrix, $|\psi^-\rangle$ is the singlet state $(|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$, and p, the mixedness parameter (lying between 0 and 1), Bob gets the maximum information about Alice's measurement outcomes given by $\mathcal{S}(\rho_{A|j}^{R_B})$ when they measure the same observables on their respective system. Hence, classical information quantifies Bob's maximum knowledge about Alice's measurement outcome in a specific direction, say in the eigen basis of $\rho_{A|j}^{R_B}$.

According to our considered game, when Alice communicates her choice, say, R_A (where 'A' labels Alice's choice), Bob measures same the observable $R_B = R_A$ on his particle (labeled by 'B'). Due to Bob's measurement, the reduced uncertainty measured by the conditional von-Neumann entropy of the state, ρ_{AR_B} given by Eq.(17) now becomes

$$\mathcal{S}(A|R_B) = \mathcal{S}(\rho_A) - C_B^R(\rho_{AB}), \tag{20}$$

where $C_B^R(\rho_{AB}) = \mathcal{S}(\rho_A) - \sum_i p_i^{R_B} \mathcal{S}(\rho_{A|i}^{R_B})$ as obtained from the Eq.(18) without taking the maximization. This is the information obtained by Bob when he makes a measurement of the observable R_B on his system. $C_B^R(\rho_{AB})$ gives the information about Alice's measurement outcomes when she measures in the eigenbasis of $\rho_{A|i}^{R_B}$ on

her particle. Bob's maximum information about Alice's measurement outcome in the eigenbasis $\rho^{R_B}_{A|i}$ is given by $C^M_B(\rho_{AB}) = \max_{R_B} C^R_B(\rho_{AB})$ which is known as the classical information where the maximization is taken over all possible observables R_B. After Bob's measurement, Alice measures the observable R_A on her particle. Now, Alice's reduced uncertainty for the measurement of observable R_A is given by

$$\mathcal{H}(R_A|R_B) = \mathcal{H}(R_A) - C^{R,R}_{A,B}(\rho_{AB}), \tag{21}$$

with

$$C^{R,R}_{A,B}(\rho_{AB}) = \mathcal{H}(R_A) - \sum_i p^{R_B}_i \mathcal{H}(q^{R_A}_i), \tag{22}$$

where $\mathcal{H}(R_A)$ is the Shannon entropy of the probability distribution $\{q^{R_B}_k\}$ corresponding to different measurement outcomes $\{k\}$ for the measurement of observable R_A on Alice's particle and $\mathcal{H}(q^{R_A}_i)$ is the Shannon entropy of the conditional probability distribution $\{q^{R_A}_{k|i}\}$ for the measurement of observable R_A on Alice's particle, given that Bob gets ith outcome for the measurement of the same observable (R_B) on his particle. We define the quantity $C^{R,R}_{A,B}(\rho_{AB})$ as the *extractable classical information.*

Similarly, when both Alice and Bob measures the observable S, the conditional entropy becomes becomes

$$\mathcal{H}(S_A|S_B) = \mathcal{H}(S_A) - C^{S,S}_{A,B}(\rho_{AB}), \tag{23}$$

where $C^{S,S}_{A,B}(\rho_{AB})$ is the *extractable classical information* for the measurement of the observable S on the both particles. Using the inequality (15) and Eqs. (21) and (23), we obtain the following uncertainty relation

$$\begin{aligned}\mathcal{H}(R_A|R_B) + \mathcal{H}(S_A|S_B) \;&\geq\; \log_2\frac{1}{c} + \mathcal{S}(\rho_A) - C^{R,R}_{A,B}(\rho_{AB}) \\ &\quad - C^{S,S}_{A,B}(\rho_{AB}),\end{aligned} \tag{24}$$

where ρ_A is the density state of Alice's particle. Now, using Fano's inequality [11], Eq.(24) becomes

$$\begin{aligned}\mathcal{H}(p^R_d) + \mathcal{H}(p^S_d) \;&\geq\; \log_2\frac{1}{c} + \mathcal{S}(\rho_A) - C^{R,R}_{A,B}(\rho_{AB}) \\ &\quad - C^{S,S}_{A,B}(\rho_{AB}),\end{aligned} \tag{25}$$

where $\mathcal{H}(p^\alpha_d)$ is the Shannon entropy of the probability distribution $\{p^\alpha_d\}$ when Alice and Bob measure same observable $\alpha \in \{R, S\}$ and get different outcomes. Eq.(25) represents our new uncertainty relation when both Alice and Bob measure two incompatible observables R and S. Hence, the lower bound of Bob's uncertainty about Alice's measurement outcomes is given by

$$\mathcal{L}_4(\rho_{AB}) = \log_2\frac{1}{c} + \mathcal{S}(\rho_A) - C^{R,R}_{A,B}(\rho_{AB}) - C^{S,S}_{A,B}(\rho_{AB}). \tag{26}$$

In the following analysis we compare this bound $\mathcal{L}_4(\rho_{AB})$ with the lower bounds through the quantum strategy obtained earlier in the literature, *viz.*, $\mathcal{L}_1(\rho_{AB})$ (given by Eq.(6)) [7, 10], the bound $\mathcal{L}_2(\rho_{AB})$ (given by Eq.(11) [12], the bound $\mathcal{L}_3(\rho_{AB})$ (given by Eq.(12)) [15], as well as the bound $\mathcal{L}_0(\rho_{AB})$ (given by Eq.(16) with $\mathcal{S}(\rho_B) = \mathcal{S}(Tr_A\rho_{AB})$) obtained with the help of the classical strategy for various classes of pure and mixed entangled and separable states. We show that the lower bound given by Eq.(26) is optimal [15] in all cases.

Pure entangled state : Here we consider a pure entangled state ρ^{PE}_{AB}, given by

$$\rho^{PE}_{AB} = \sqrt{\alpha}|01\rangle_{AB} - \sqrt{1-\alpha}|10\rangle_{AB}, \tag{27}$$

where α lies between 0 and 1, and the state ρ^{PE}_{AB} is maximally entangled for $\alpha = \frac{1}{2}$. The classical information (when Alice measures her particle) is given by

$$C^M_B(\rho^{PE}_{AB}) = \mathcal{H}(\alpha), \tag{28}$$

where $\mathcal{H}(\alpha) = \alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha)$. $C^M_B(\rho^{PE}_{AB})$ gives the information about Alice's measurement outcome in the direction $\{\mu\cos[\phi_S]\sin[\theta_S], \mu\sin[\phi_S]\sin[\theta_S], \frac{1-2\alpha+\cos[\theta_S]}{1+\cos[\theta_S]-2\alpha\cos[\theta_S]}\}$ (where $\mu = \frac{2\sqrt{\alpha(1-\alpha)}}{1+\cos[\theta_S]-2\alpha\cos[\theta_S]}$) to Bob when he measures along $\{\sin(\theta_S)\cos(\phi_S), \sin(\theta_S)\sin(\phi_S), \cos(\theta_S)\}$. Let us consider that before playing the game Alice and Bob discuss about their strategy, such as, choices of the state and measurement settings. Alice chooses those settings for which Bob's uncertainty about her measurement outcome will be minimum as well as maximize the lower bound of uncertainty in the classical strategy (given by Eq.(16)), i.e., $\log_2\frac{1}{c} = 1$. With the help of the fine-grained uncertainty relation [15, 16], one can obtain the winning probability (corresponding to minimum uncertainty) when Alice and Bob both measure the same observable and get different outcomes, i.e., $a \oplus b = 1$ [15]. The winning probability is given by

$$\begin{aligned}P^{\text{game}}(\rho^{PE}_{AB}) = \;&\frac{1}{4}(3 + 2\sqrt{\alpha(1-\alpha)} \\ &+ (1 - 2\sqrt{\alpha(1-\alpha)})\cos[2\theta_S])\end{aligned} \tag{29}$$

Bob's uncertainty about Alice's outcome would be minimum for the choice of observables given by

$$\begin{aligned}R &= \sigma_z \qquad (\text{i.e.,}\;\; \theta_S = 0), \\ S &= \sigma_x \qquad (\text{i.e.,}\;\; \theta_S = \frac{\pi}{2}),\end{aligned} \tag{30}$$

which leads to $p^R_d = 1$, and the infimum probability $p^S_{\text{inf}} = 1/2 + \sqrt{\alpha(1-\alpha)}$. Hence, the optimal lower bound obtained from the Eq.(12) is given by [15]

$$\mathcal{L}_3(\rho^{PE}_{AB}) = \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}). \tag{31}$$

When Bob chooses the *classical strategy*, he first prepares two copies of the state $Tr_B(\rho_{AB}^{PE})$ and send one to Alice. For the above choice of observables, Bob's uncertainty about Alice's measurement outcome is maximum (with respect to the measurement settings, i.e., $\log_2 \frac{1}{c} = 1$), and is given by the inequality (15). The lower bound (16) is given by

$$\mathcal{L}_0(\rho_{AB}^{PE}) = 1 + \mathcal{H}(\alpha). \tag{32}$$

In the *quantum strategy* using the uncertainty relations proposed in Refs. [7] and [12], Bob's uncertainty is lower bounded by ((6) and (11)), respectively, which turn out to be equal, given by

$$\mathcal{L}_1(\rho_{AB}^{PE}) = \mathcal{L}_2(\rho_{AB}^{PE}) = 1 - \mathcal{H}(\alpha). \tag{33}$$

However, in practice Bob is unable to reduce his uncertainty upto the above level, since $\mathcal{L}_1(\rho_{AB}^{PE})$ is not the optimal lower bound. The main reason is that Bob only extracts the information $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{PE})$ ($C_{A,B}^{\sigma_x,\sigma_x}(\rho_{AB}^{PE})$) given by $\mathcal{H}(\alpha)$ ($1 - \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)})$) when both of them measure the same spin observables $R = \sigma_z$ ($S = \sigma_x$) on their respective particle. Hence, the lower bound (given by Eq.(26)) of Bob's uncertainty is given by

$$\mathcal{L}_4(\rho_{AB}^{PE}) = \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}). \tag{34}$$

From the Eqs.(31) and (34), it is clear that the quantities $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{PE})$ and $C_{A,B}^{\sigma_x,\sigma_x}(\rho_{AB}^{PE})$ are responsible for reducing Bob's uncertainty about Alice's measurement outcome optimally. This explains in terms of physical resources why the lower bound $\mathcal{L}_1(\rho_{AB}^{PE})$ ($\leq \mathcal{L}_3(\rho_{AB}^{PE})$) is not experimentally reachable, whereas the lower bound $\mathcal{L}_3(\rho_{AB}^{PE})$ given operationally by fine-graining is indeed attainable.

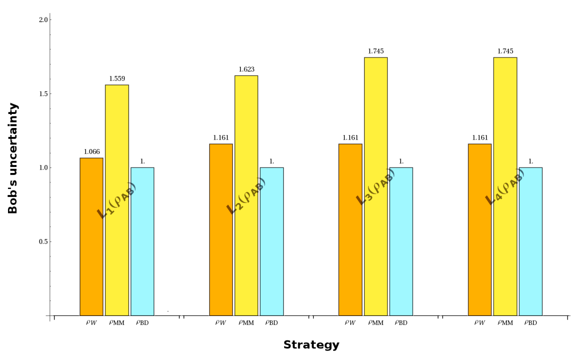

FIG. 1: A comparison of the different lower bounds for the (i) Werner state with $p = 0.723$, (ii) the state with maximally mixed marginals with the ci's given by $c_x = 0.5$, $c_y = -0.2$, and $c_z = -0.3$, and (iii) the Bell diagonal state with $p = 0.5$.

Werner State : For the class of Werner State ρ_{AB}^W, given by Eq.(19), the classical information is given by

$$C_B^M(\rho_{AB}^W) = 1 - \mathcal{H}(\frac{1-p}{2}). \tag{35}$$

$C_B^M(\rho_{AB}^W)$ gives Bob the information about the measurement outcome of Alice when they measure same observables. The quantum discord of the state ρ_{AB}^W is given by

$$D_B(\rho_{AB}^W) = \mathcal{I}(\rho_W) - C_B^M, \tag{36}$$

where $\mathcal{I}(\rho_{AB}^W) = 2 + 3\frac{1-p}{4}\log_2\frac{1-p}{4} + \frac{1+3p}{4}\log_2\frac{1+3p}{4}$ is the mutual information of ρ_{AB}^W.

In the *Classical strategy*, for the choice observables given by Eq.(30) (which minimize Bob's uncertainty optimally [15]), Bob's uncertainty is lower bounded by (16)

$$\mathcal{L}_0(\rho_{AB}^W) = 2, \tag{37}$$

where $\rho_A^W = Tr_B[\rho_{AB}^W] = \frac{I}{2}$. When Bob uses the *quantum strategy* [7, 10], his uncertainty (given by Eq.(5)) is bounded by

$$\mathcal{L}_1(\rho_{AB}^W) = 2 - \mathcal{I}(\rho_{AB}^W), \tag{38}$$

where for the state ρ_{AB}^W, $\mathcal{S}(A|B) = 1 - \mathcal{I}(\rho_{AB}^W)$.The improved lower bound (11) given by Pati et al.[12] for the Werner class of states turns out to be

$$\mathcal{L}_2(\rho_{AB}^W) = 2 - 2C_B^M(\rho_{AB}^W) = 2\mathcal{H}(\frac{1-p}{2}) \tag{39}$$

Note that Bob is able to gain his knowledge about Alice's measurement outcomes by an amount $C_B^M(\rho_{AB}^W)$ when both Alice and Bob measure the same observables $R = \sigma_z$ ($S = \sigma_x$) on their respective particles. Hence, Bob's uncertainty (given by Eq.(25)) is lower bounded by

$$\mathcal{L}_3(\rho_{AB}^W) = 2 - 2C_B^M(\rho_{AB}^W) = 2\mathcal{H}(\frac{1-p}{2}). \tag{40}$$

Now, using fine-graining the optimal lower bound for Bob's uncertainty is given by [15]

$$\mathcal{L}_4(\rho_{AB}^W) = 2\mathcal{H}(\frac{1-p}{2}), \tag{41}$$

Thus, for the Werner class of states, Bob can operationally minimize his uncertainty about Alice's measurement outcome upto $2\mathcal{H}(\frac{1-p}{2})$ (given by Eqs.(39) ,(40) and (41)). The lower bound $\mathcal{L}_1(\rho_{AB}^W)$ ($\leq \mathcal{L}_3(\rho_{AB}^W)$) is not experimentally reachable.

Bell diagonal state : The Bell diagonal state, used in Ref.[10] is given by

$$\rho_{AB}^{BD} = p\rho_2 + (1-p)\rho_S, \tag{42}$$

where ρ_2 is the density matrix of the state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$. The classical information of the state ρ_{AB}^{BD} is given by

$$C_B^M(\rho_{AB}^{BD}) = 1. \tag{43}$$

Here $C_B^M(\rho_{AB}^{BD})$ gives Bob the information about Alice's measurement outcome for the measurement

along $\{\sin(\theta_S)\cos(\phi_S), -\sin(\theta_S)\sin(\phi_S), \cos(\theta_S)\}$ from his measurement outcome along the direction $\{\sin(\theta_S)\cos(\phi_S), \sin(\theta_S)\sin(\phi_S), \cos(\theta_S)\}$. The quantum discord of ρ_{AB}^{BD} is given by

$$D_B(\rho_{AB}^{BD}) = 1 - \mathcal{H}(p), \tag{44}$$

where $\mathcal{I}(\rho_{AB}^{BD})$ $(= 2 - \mathcal{H}(p))$ is the mutual information of the state ρ_{AB}^{BD}. From the Eqs.(43) and (44), it is clear that for the state ρ_{AB}^{BD}, $C_B^M(\rho_{AB}^{BD}) \geq D_B(\rho_{AB}^{BD})$.

In the *classical strategy* for the choice of above observables, Bob's uncertainty is bounded by

$$\mathcal{L}_0(\rho_{AB}^{BD}) = 2. \tag{45}$$

In the *quantum strategy*, theoretically Bob's uncertainty (obtained using Eq.(3) and (7)) is lower bounded by

$$\mathcal{L}_1(\rho_{AB}^{BD}) = \mathcal{L}_2(\rho_{AB}^{BD}) = \mathcal{H}(p), \tag{46}$$

where $S(A|B) = \mathcal{H}(p) - 1$. With the measurement on his system of an observable communicated by Alice, Bob extracts the classical information by an amount $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{BD}) = 1 - \mathcal{H}(p)$ ($C_{A,B}^{\sigma_y,\sigma_y}(\rho_{AB}^{BD}) = 1$) for the spin measurement along $z-$ direction ($y-$ direction). Hence, Bob's uncertainty is lower bounded by

$$\mathcal{L}_4(\rho_{AB}^{BD}) = \mathcal{H}(p). \tag{47}$$

In this case the optimal lower bound for Bob's uncertainty about Alice's measurement outcome given by fine-graining [15] also turns out to be

$$\mathcal{L}_3(\rho_{AB}^{BD}) = \mathcal{H}(p). \tag{48}$$

Here the lower bound predicted by [7, 12] is optimal. Eqs.(47) and (48) show that the extactable classical information $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{BD}) = 1 - \mathcal{H}(p)$, $C_{A,B}^{\sigma_y,\sigma_y}(\rho_{AB}^{BD}) = 1$ is responsible for reducing Bob's uncertainty optimally.

Maximally mixed marginal state : The maximally mixed marginal state ρ_{AB}^{MM} is given by

$$\rho_{AB}^{MM} = \frac{1}{4}(\mathcal{I} + \sum_{i=x,y,z} c_i \sigma_i \otimes \sigma_i). \tag{49}$$

where the coefficients c_i's ($i \in \{x, y, z\}$) are constrained by the eigenvalues $\lambda_i \in [0, 1]$ of ρ_{AB}^{MM} given by

$$\begin{aligned}
\lambda_0 &= \frac{1 - c_x - c_y - c_z}{4} \\
\lambda_1 &= \frac{1 - c_x + c_y + c_z}{4} \\
\lambda_2 &= \frac{1 + c_x - c_y + c_z}{4} \\
\lambda_3 &= \frac{1 + c_x + c_y - c_z}{4}
\end{aligned} \tag{50}$$

The mutual information of the state ρ_{AB}^{MM} is given by

$$\mathcal{I}(\rho_{AB}^{MM}) = 2 + \sum_{j=0}^{3} \lambda_j \log 2[\lambda_j]. \tag{51}$$

The classical information of the sate is given by [21]

$$\begin{aligned}
C_B^M(\rho_{AB}^{MM}) = & \frac{1 - c_M}{2} \log 2[1 - c_M] \\
& + \frac{1 + c_M}{2} \log 2[1 + c_M],
\end{aligned} \tag{52}$$

where $c_M = \max[|c_x|, |c_y|, |c_z|]$, and the quantum discord of the state ρ_{AB}^{MM} is given by

$$D_B(\rho_{AB}^{MM}) = \mathcal{I}(\rho_{AB}^{MM}) - C_B^M(\rho_{AB}^{MM}). \tag{53}$$

As usual, before playing the game, Alice and Bob discuss the measurement settings (i.e., strategy for the game) for the shared state ρ_{AB}^{MM}. To optimize the uncertainty, Bob takes the help of the fine-grained uncertainty relation (FUR) [15]. Here, the winning probability when Bob Alice and Bob both measure the observable S is given by

$$\begin{aligned}
P_S^{\text{game}} = & \frac{1}{2}(1 - c_x \sin^2[\theta_S] \cos^2[\phi_S] \\
& - c_y \sin^2[\theta_S] \sin^2[\phi_S] - c_z \cos^2[\theta_S]).
\end{aligned} \tag{54}$$

The measurement settings may be chosen such that the quantity P_S^{game} is maximized. For the measurement setting σ_z (i.e., $\theta_S = 0$), $P_{\sigma_z}^{\text{game}}$ will be maximum when $c_z - c_x < 0$. Here we consider $c_x = 0.5$, $c_y = -0.2$, and $c_z = -0.3$, and for this choices, the observable $R = \sigma_z$ and $S = \sigma_x$ minimizes Bob's uncertainty [15]. For the above choice the optimal lower bound of Bob's uncertainty is lower bounded by[15]

$$\mathcal{L}_3(\rho_{AB}^{MM}) \approx 1.745. \tag{55}$$

When Bob chooses the classical strategy, for the above choice of observable his uncertainty given by (15) is lower bounded by

$$\mathcal{L}_0(\rho_{AB}^{MM}) = 2, \tag{56}$$

Employing the quantum strategy, Bob's uncertainty (5) is lower bounded by

$$\mathcal{L}_1(\rho_{AB}^{MM}) \approx 1.5589, \tag{57}$$

whereas, the bound [12] is given by

$$\mathcal{L}_2(\rho_{AB}^{MM}) \approx 1.6226, \tag{58}$$

where the classical information $C_B^M(\rho_{AB}^{MM}) = 0.1887$ (obtained from Eq.(52) using our choice of c_i's) and the quantum discord $D_B(\rho_{AB}^{MM}) = 0.2524$ (obtained from Eq.(53)), tightens Berta's lower bound given by Eq.(57)[7].

When Both Alice and Bob measure same observable, σ_z (σ_x) on their respective system, Bob extracts the information given by $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{MM}) = 0.0659$ ($C_{A,B}^{\sigma_x,\sigma_x}(\rho_{AB}^{MM}) = 0.1887$) for the above choice of c_i's. Now, using Eq.(25) Bob's uncertainty is lower bounded by

$$\mathcal{L}_4(\rho_{AB}^{MM}) \approx 1.745, \tag{59}$$

which is equal to the optimal lower bound obtained using fine-grained uncertainty relation [15]. One sees that though in this case, $\mathcal{L}_2(\rho_{AB}^{MM})$ tightens the bound $\mathcal{L}_1(\rho_{AB}^{MM})$, both of them are unable to be realized operationally. Fig.1 depicts the main result of the paper, *viz.*, the optimal lower bound obtained through the quantum strategy where the concept of *extractable classical information* is applied.

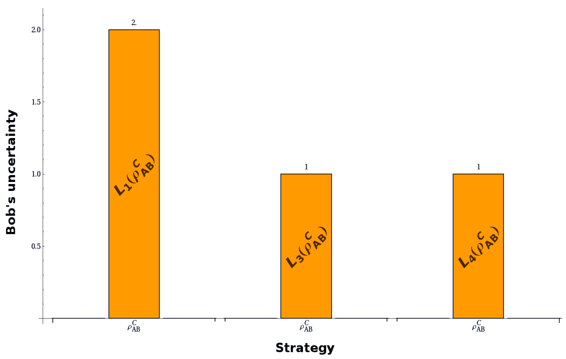

FIG. 2: A comparison of the different lower bounds for the shared classical state choosing p=0.5.

Classical state : Now, we consider classical state ρ_{AB}^C, given by

$$\rho_{AB}^C = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|. \tag{60}$$

The state, ρ_{AB}^C is a zero discord state [18], i.e.,

$$D_B(\rho_{AB}^C) = 0. \tag{61}$$

The classical information of the state ρ_{AB}^C is given by

$$C_B^M(\rho_{AB}^C) = \mathcal{H}(p). \tag{62}$$

$C_B^M(\rho_{AB}^C)$ gives the information about Alice's measurement outcome for the measurement of observable σ_z to Bob, when Bob measures the same observable σ_z. The winning probability of the game characterized by winning condition $a \oplus b = 1$ [15] is

$$P^{\text{game}}(\rho_{AB}^C) = \frac{\sin^2[\theta_S]}{2}. \tag{63}$$

Hence, the choices (for Alice) of the set of observables $\{R, S\}$ (which minimize Bob's uncertainty about Alice's outcome) are given by Eq.(30).

In this case, when Bob chooses the *classical strategy*, his uncertainty (given in Eq.(3)) for the choices of settings given by Eq.(30) is lower bounded by an amount

$$\mathcal{L}_0(\rho_{AB}^C) = 1 + \mathcal{H}(p). \tag{64}$$

When Bob applies the *quantum strategy* [7, 10, 12] his uncertainty is lower bounded by

$$\mathcal{L}_1(\rho_{AB}^C) = \mathcal{L}_2(\rho_{AB}^C) = 1. \tag{65}$$

For the state ρ_{AB}^C, Bob's *extractable* classical information (given by Eq.22) is $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{MM}) = \mathcal{H}(p)$ ($C_{A,B}^{\sigma_x,\sigma_x}(\rho_{AB}^{MM}) = 0$) when both of them measure the same observable $R = \sigma_z$ ($S = \sigma_x$) on their respective particles. Hence, the lower bound given by Eq.(26) becomes

$$\mathcal{L}_4\rho_{AB}^C = 1, \tag{66}$$

Finally, the optimal lower bound given by Eq.(12) is also

$$\mathcal{L}_3\rho_{AB}^C = 1. \tag{67}$$

Hence in this case, $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \mathcal{L}_4 = 1 < \mathcal{L}_0$. We thus observe that even purely classical correlations can play a role in reducing the uncertainty using a shared bipartite state when the quantum strategy is employed. This result is displayed in Fig.2.

To summarize, in the present work we derive a new uncertainty principle which enables to reduce the uncertainty of Bob about Alice's measurement outcome with the help of a shared state and communication between the two. We introduce the physical quantity, called *extractable classical information* and show that it is responsible for *optimally* reducing the uncertainty for the measurement of two non-commuting observables. The lower bound of our proposed uncertainty relation is equal to the *optimal* lower bound obtained with the help of the fine-grained uncertainty relation [15, 16], as we show for pure and mixed states with quantum and classical correlations. Our analysis further explains how the uncertainty may be reduced using the quantum strategy even in the absence of quantum correlations when the two parties share just a classically correlated state.

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